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COMPUTATIVE FXAMINATION OF BENDING STRENGTH OF GIRDERS

ORIGINALLY CURVED AND SUBJECTED TO LONGITUDINAL COMPRESSION.

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COMPUTATIVE EXAMINATION OF BENDING STRENGTH OF GIRDERS ORIGINALLY CURVED AND SUBJECTED TO LONGITUDINAL COMPRESSION.*

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The 1918 military specifications (BLV) contained the following paragraphs:

"It must not only be demonstrated that the longitudinal stress generated in a girder is below the Euler compression force, but also that, with an initial deflection equal to 1/200 of the length of the girder, the stress produced in it by half the prescribed breaking load, does not exceed half the breaking strength.

"Short compression struts, whose cross-section may have a radius of gyration i, must in so far as s/i<105 for cast iron and steel and s/i<110 for wood, not be computed according to the Euler, but according to the Tetmajer compression formula.

This stipulation is based on an article by Müller Breslau.** It is accordingly requested that at least double the actual load on the lever arm, 1/200 of the length of the girder, should at most create a stress near the limit of proportionality.

Since it is difficult to determine the actual load in airplane construction and since the introduction of the breaking load into the strength computation was customary, the Müller Breslau requirements, of double the actual load and the limit of proportionality, were converted into the stipulations of half the breaking load and half the breaking strength.

The following consideration is based on the designations in

Fig. 1. The girder s(cm) long has an incipient parabolic curve indicated by the expression

$$f = 4 \, r \, s \left[\frac{x}{s} - \left(\frac{x}{s} \right)^2 \right] \tag{1}$$

The curve is the greatest in the middle and has the value $f_{max} = r \, s$.

Under the influence of the longitudinal forces S(kg), the girder is bent to the position of the dash line. At the point P, with the coordinates x and y + f, there is a bending moment

$$M = S \, (y + f) \tag{2}$$

This moment is held in equilibrium by the elastic forces of the girder.

$$M = - E \, I \, \frac{d^2 y}{d x^2} \tag{3}$$

in which E (kg/cm²) is the modulus of elasticity and I (cm⁴) the moment of inertia of the girder.

Expressions 1, 2 and 3, after the introduction of

$$k = \sqrt{\frac{E \, I}{S}} \tag{4}$$

are combined in the differential equation

$$k^2 \frac{d^2 y}{d x^2} + y = 4 \, r \, s \left[\left(\frac{x}{s} \right)^2 - \frac{x}{s} \right] \tag{5}$$

This is integrated in the usual manner, as also, after introduction of the integration constants for x = o and x = s and the simplification

$$\frac{s}{k} = \sqrt{\frac{s^2 \, S}{E \, I}} = \alpha \qquad\qquad (6)$$

The solution:

$$y = 4 \, r \, s \, \left\{ \frac{2}{\alpha^2} \left[\cos \frac{x}{k} + \frac{(1 - \cos \alpha)}{\sin \alpha} \sin \frac{x}{k} - 1 \right] + \left[\left(\frac{x}{s}\right)^2 - \frac{x}{s} \right] \right\} \; (7)$$

The bending moment from equation (2) is transformed to

$$M = \frac{8 \, r \, s}{\alpha^2} \, S \left[\cos \frac{x}{k} + \frac{(1 - \cos \alpha)}{\sin \alpha} \sin \frac{x}{k} - 1 \right] \qquad (8)$$

and consequently increases proportionally with the maximum initial deflection $r \, s$.

This bending moment attains, with $x = 1/2$, its maximum value

$$M_{max} = 8 \, r \, s \left[\frac{1}{\cos \frac{\alpha}{2}} - 1 \right] \frac{E \, I}{s^2} \qquad (9)$$

On the other hand

$$M_{max} = \sigma \, W \qquad\qquad (10)$$

in which $\sigma \, (kg/cm^2)$ represents the bending stress in the middle of the girder and $W \, (cm^3)$ the resistance moment of the girder.

According to the already cited stipulation of the 1918 military specifications, the bending stress under the action of the mth part of the bending load as longitudinal load, must not exceed a certain n^{th} part of the breaking stress. From this requirement follows

$$\sigma = \frac{K_b}{n} - \frac{S_k}{mF} \qquad\qquad (11)$$

in which the designations K_b (kg/cm²), for the breaking bending stress; S_k (kg), for the Euler bending load and F (cm³), for the cross-sectional area of the girder, are newly introduced.

By employing the Euler equation

$$\frac{S_k}{m} = \frac{\pi^2 E\ I}{m\ s^2} \tag{12}$$

in equation (6) we have

$$\alpha = \frac{\pi}{\sqrt{m}} \tag{13}$$

By combining equations (9) and (10) and introducing equations (12) and (13), we have

$$W\left[\frac{K_b}{n} - \frac{\pi^2 E\ I}{m\ s^2}\right] = 6\ r\ s\left[\frac{1}{\cos\ \dfrac{\pi}{2\sqrt{m}}} - 1\right]\frac{E\ I}{s^2} \tag{14}$$

whence we obtain, for the length s of the girder, the quadratic equation

$$s^2 - 8\ r\ n\ \frac{E\ I}{K_b\ W}\left[\frac{1}{\cos\ \dfrac{\pi}{2\sqrt{m}}} - 1\right]s = \pi^2\ \frac{n\ E\ I}{m\ K_b\ F} \tag{15}$$

If D (cm) designates the distance between the outermost fibers of the cross-section of the bent girder under consideration, we may substitute

$$\frac{I}{W} = \frac{D}{2} \tag{16}$$

If we make the radius of gyration i, through the equation
i = e D dependent on D, we may also substitute $\qquad$ (17)

$$\frac{i}{F} = i^2 = e^2 \ D^2 \tag{18}$$

By employing the simplified equations (16) and (18), we finally obtain

$$\frac{s}{D} = A + \sqrt{A^2 + B^2} \tag{19}$$

in which are introduced

$$A = 2 \ r \ n \ \frac{E}{K_b} \left[\frac{1}{\cos \ \dfrac{\pi}{2 \sqrt{m}}} - 1 \right] \tag{20}$$

and

$$B = \pi e \sqrt{\frac{n \ E}{m \ K_b}} \tag{21}$$

The quantity A is independent of the cross-section of the girder and increases in direct proportion with the quantities r and n and decreases with increasing values of m. The quantity B increases with increasing n and decreasing m.

This important result means that, for a given cross-section of a girder, designated by the factor e, for a fixed ratio of E to K_b, as well as for chosen quantities, r, m, and n, there i a single ratio s/D, which fulfills the requirement that, with an initial maximum deflection r s of the s-long girder for the m part of the Euler bending load, also the nth part of the breakin stress will be attained.

Since the value s/D, thus computed for any cross-section, applies to all cross-sections possessing geometrically similar or like e values, it constitutes a great simplification, as will

subsequently be demonstrated in the example for cylindrical pipes.

The stress produced by the bending load S_k is given by equation

$$\sigma_k = \frac{S_k}{F} = \left(\pi e \frac{D}{s}\right)^2 E \qquad (22)$$

Accordingly, for a fixed ratio s/D, also the stress produced by the bending load remains the same for all similar cross-section

Computation example. Cylindrical tubes.— For the cross-section of a cylindrical tube the equations

$$1 = \frac{\pi}{64} \left(D^4 - d^4\right)$$

and

$$F = \frac{\pi}{4} \left(D^2 - d^2\right).$$

apply, when D (cm) designates the outside and d (cm) the inside diameter of the tube, hence:

$$e = \frac{1}{4}\sqrt{\left[1 + \left(\frac{d}{D}\right)^2\right]}$$

In Fig. 2, e and d/D are plotted against each other.

After choosing the ratio $E/K_b = 550$, which corresponds approximately to the malleable, seamless steel tubes employed in airplane construction, the ratio s/D is plotted against the quantity r (Fig. 3) for the quantities m = n = 1.8, m = n = 2.0 and m = n = 2.2, as well for solid rods as for tubes with infinitely thin walls

The curves of Fig. 3 indicate that the influence of the thic

ness of the walls vanishes for high values of r, i.e. large initial
deflections of the girder. For a very small initial deflection,
hence small values of r, the influence of the thickness of the
wall cannot be disregarded. If the initial deflection is taken too
small, we obtain s/D ratios, which lie beyond the region of appli-
cation of the Euler bending formula for which the above considera-
tion alone holds good. The Euler-Tetmajer-limits are introduced in
to Fig. 3, in accordance with the ratio which applies to cast iron
and steel (See paragraph 2 of the quotation from the 1918 BLV at
the beginning of this treatise).

$$\frac{s}{D} \gtreqqless \frac{110}{4} \sqrt{\left[1 + \left(\frac{d}{D} \right)^2 \right]}$$

Solid cylindrical rod

$$\frac{s}{D} \gtreqqless 27.5$$

Tube with infinitely thin walls

$$\frac{s}{D} \gtreqqless 38.9$$

In the 1918 BLV, the stipulated quantity r = 1/200 = 0.005
gives values of s/D which lie barely above the Tetmajer limits.

If we examine the struts of airplanes which have given satis-
factory service, we find no simple illustration of the distribution
of the s/D values. Many of them lie in the Tetmajer and others
far within the Euler region.

Since with slender wing struts, bucklings are not infreque

as a consequence of some injury and since such struts have greate

s/D values than would correspond to an initial buckling of 1/200

or r = 0.005, a departure from the 1918 BLV is justified.

The ordinarily employed seamless steel tubes have a diameter

ratio of d/D = 0.9. For this value, as also for the quantities

r = 0.01 and m = n = 2, we have

$$\frac{s}{D} = 65.$$

The stress σ_k for a cylindrical tube is found to be

$$\sigma_k = \frac{\pi^2}{16}\left[1 + \left(\frac{d}{D}\right)^2\right]\left(\frac{D}{s}\right)^2 E$$

For E = 2150000 kg/cm², for d/D = 0.9 and for σ/D = 65,

we have

$$\sigma_k = 570 \ kg/cm^2$$

If, as is usually the case, s/D is not computed for

r = 0.01 and m = n = 2, but is differently employed, then it

must be ascertained at what share of the bending load the desired

nth part of the breaking stress occurs. For this purpose, for

fixed values r and n with equations (19), (20), and (21), the

m value must be found at which equation (19) is fulfilled.

In Fig. 4 m and s/D are plotted against each other for the

fixed values E/K_b = 550, d/D = 0.9, r = 0.01, and n = 2.

For s/D = 65, corresponding to the previous computation,

m = 2. For smaller values of s/D, a larger value of m must b

used, for instance, the value m = 2.9 corresponds to s/D = 40.

The application of the 1918 BLV results, therefore, in a lesser utilization of the resistance to axial compression in small girders than in large ones. In the computation, this circumstance is based on the bending strength.

The results explained in the example for cylindrical tubes may also be found in a similar way for other materials and oth: cross-sections. Here it should be remembered that the value n is specially adapted to the material. For girders with a high elasticity limit, $n < 2$. For wood, however, $n = 2$ is recommended.

Conclusion. - We examined the stipulation contained in the 1918 BLV, that a girder subjected to longitudinal compression under the influence of half of the specified breaking load, along with the Euler bending safety with an initial deflection of 1/200 of the length of the girder, can, at the most, be subjected to half the stipulated breaking stress. We found a generally applicable ratio for the relation of the length s(cm) of the girder to the distance D (cm) between the outermost fibers of the cross-section subjected to bending. This ratio gives, for all geometrically similar girder cross-sections (also $e = i/D$ constant), a length limit fulfilling these conditions.

By means of an example, we showed that the stipulation of the initial deflection of 1/200 of the girder length is not fulfilled by compression struts in use, but that, on the contrary, the assumption of a greater initial deflection, e.g. 1/100 is advisable.

Since the 1818 BLV stipulation is only fulfilled for a single value of s/D, we showed in what manner the necessary breaking portion of the bending load is found for the values of s/D differing from said value.

Translated by the National Advisory Committee for Aeronautics.

Fig. 1

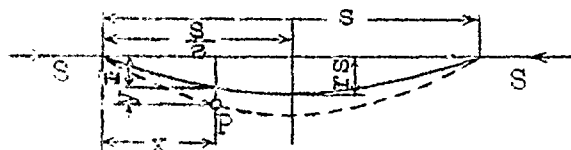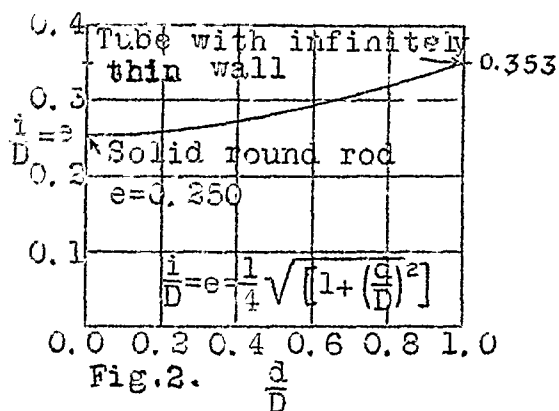$$\frac{E}{K_b}=550 \quad r=0.01 \quad n=3 \quad \frac{d}{D}=0.9$$

$$A=22\left[\frac{1}{\cos\dfrac{\pi}{2\sqrt{m}}}-1\right]$$

$$B=35\sqrt{\frac{1}{m}}$$

$$\frac{s}{D}=A+\sqrt{A^2+B^2}$$

Fig.2. $\frac{d}{D}$

Tube with infinitely thin wall — 0.353

Solid round rod e=0.250

$$\frac{i}{D}=e=\frac{1}{4}\sqrt{\left[1+\left(\frac{d}{D}\right)^2\right]}$$

$\frac{s}{D}$

Fig. 4

$\frac{s}{D}$

m=n=2.2   m=n=2.0   m=n=1.8

$$r=\frac{1}{200}=0.005$$

Solid round rod

Euler
Tetmajer
Limit

Tube with infinitely thin wall

Fig.3.

$\frac{s}{D}$